PROOF OF A PARTITION IDENTITY CONJECTURED BY LASSALLE

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Abstract. We prove a partition identity conjectured by Lassalle (Adv. in Appl. Math. **21** (1998), 457–472).

The purpose of this note is to prove the theorem below which was conjectured by Lassalle [1, 2]. In order to state the theorem, we introduce the following notations. Let $(a)_n = a(a+1)\cdots(a+n-1)$. For a partition μ of n let the length $l(\mu)$ be the number of the parts of μ, m_i the number of parts i, $z_\mu = \prod_{i\geq 1} i^{m_i(\mu)} m_i(\mu)!$ and $\left\langle {\mu \atop r} \right\rangle$ the number of ways to choose r different cells from the diagram of the partition μ taking at least one cell from each row. Then the following theorem holds for $n \geq 1$.

Theorem 1.

$$\sum_{|\mu|=n} \left\langle {\mu \atop r} \right\rangle \frac{X^{l(\mu)-1}}{z_\mu} \sum_{i=1}^{l(\mu)} (\mu_i)_s$$
$$= (s-1)! \binom{n+s-1}{n-r} \left[\binom{X+r+s-1}{r} - \binom{X+r-1}{r} \right] \tag{1}$$

Proof. We first observe that $\prod_{i\geq 1} i^{m_i(\mu)} = \prod_{i=1}^{l(\mu)} \mu_i$ and that $\frac{l(\mu)!}{m_1!\cdots m_n!}$ is the number of compositions of n which are permutations of the parts of μ. Let us denote this number by $C(\mu)$. After division by $s!$ the left-hand side can be rewritten as

$$\begin{aligned} \frac{\text{LHS}}{s!} &= \sum_{|\mu|=n} C(\mu) \left\langle {\mu \atop r} \right\rangle \frac{X^{l(\mu)-1}}{l(\mu)! \prod_{i=1}^{l(\mu)} \mu_i} \sum_{i=1}^{l(\mu)} \binom{\mu_i+s-1}{s} \\ &= \sum_{l=1}^{\infty} \sum_{\substack{\mu_1+\dots+\mu_l=n \\ \mu_j \geq 1}} \frac{X^{l-1}}{l!\mu_1\cdots\mu_l} \left\langle {\mu \atop r} \right\rangle \sum_{i=1}^{l} \binom{\mu_i+s-1}{s} \end{aligned}$$

For the composition μ, $\left\langle {\mu \atop r} \right\rangle$ counts the ways of choosing r points in the diagram of the composition. If we choose r_i points from part μ_i, there are $\prod_{i=1}^{l} \binom{\mu_i}{r_i}$ possible choices. Summing over all possible compositions $r = r_1 + \cdots + r_l$, where every part is ≥ 1 gives $\left\langle {\mu \atop r} \right\rangle$. Thus we get for the left-hand side of (1)

$$\frac{\text{LHS}}{s!} = \sum_{l=1}^{\infty} \sum_{\substack{\mu_1+\dots+\mu_l=n \\ \mu_j \geq 1}} \frac{X^{l-1}}{l!} \sum_{\substack{r_1+\dots+r_l=r \\ r_j \geq 1}} \frac{1}{r_1\cdots r_l} \binom{\mu_1-1}{r_1-1} \cdots \binom{\mu_l-1}{r_l-1} \sum_{i=1}^{l} \binom{\mu_i+s-1}{s}$$

It is easy to see that $\binom{\mu_i+s-1}{\mu_i-1}\binom{\mu_i-1}{r_i-1} = (-1)^{r_i-1}\binom{-s-1}{r_i-1}\binom{\mu_i+s-1}{r_i+s-1}$. Now we can evaluate the sum over the μ_j by repeated application of the Chu-Vandermonde summation formula:

$$\sum_{\mu_1+\cdots+\mu_l=n} \binom{\mu_1-1}{r_1-1}\cdots\binom{\mu_l-1}{r_l-1}\binom{\mu_i+s-1}{s} = (-1)^{r_i-1}\binom{-s-1}{r_i-1}\binom{n+s-1}{r+s-1}.$$

Thus, we get for the left-hand side of (1)

$$\frac{\mathrm{LHS}}{s!} = \sum_{l=1}^{\infty}\frac{X^{l-1}}{l!}\sum_{\substack{r_1+\cdots+r_l=r\\ r_j\geq 1}}\frac{1}{r_1\cdots r_l}\sum_{i=1}^{l}(-1)^{r_i-1}\binom{-s-1}{r_i-1}\binom{n+s-1}{r+s-1}. \tag{2}$$

The factor $\binom{n+s-1}{r+s-1} = \binom{n+s-1}{n-r}$ can be taken outside of all the sums. By comparison of (1) and (2), we see that it remains to prove

$$\sum_{l=1}^{\infty}\frac{X^{l-1}}{l!}\sum_{\substack{r_1+\cdots+r_l=r\\ r_j\geq 1}}\frac{1}{r_1\cdots r_l}\sum_{i=1}^{l}(-1)^{r_i-1}\binom{-s-1}{r_i-1}$$
$$= \frac{1}{s}\left[\binom{X+r+s-1}{r} - \binom{X+r-1}{r}\right]. \tag{3}$$

This can be done by using generating functions. We multiply both sides of the equation by Φ^r and sum over all $r \geq 0$. The right-hand side can be evaluated by the binomial theorem and gives

$$\frac{1}{s}\left((1-\Phi)^{-X-s} - (1-\Phi)^{-X}\right). \tag{4}$$

For the left-hand side we need the power series expansion of the logarithm and the equation

$$\sum_{r_i=1}^{\infty}\binom{r_i+s-1}{s}\frac{\Phi^{r_i}}{r_i} = \frac{1}{s}((1-\Phi)^{-s}-1),$$

which can be derived from the binomial theorem. So the generating function corresponding to the left-hand side of (4) evaluates as follows:

$$\sum_{l=1}^{\infty}\frac{X^{l-1}}{l!}\sum_{r_1=1}^{\infty}\frac{\Phi^{r_1}}{r_1}\sum_{r_2=1}^{\infty}\frac{\Phi^{r_2}}{r_2}\cdots\sum_{r_l=1}^{\infty}\frac{\Phi^{r_l}}{r_l}\sum_{i=1}^{l}\binom{r_i+s-1}{s}$$
$$\begin{aligned}
&=\sum_{l=1}^{\infty}\frac{X^{l-1}}{l!}\sum_{i=1}^{l}\left(\log\frac{1}{1-\Phi}\right)^{l-1}\frac{1}{s}\left((1-\Phi)^{-s}-1\right)\\
&=\frac{1}{s}((1-\Phi)^{-s}-1)\sum_{l=1}^{\infty}\frac{\left(X\log\frac{1}{1-\Phi}\right)^{l-1}}{(l-1)!}\\
&=\frac{1}{s}((1-\Phi)^{-s}-1)e^{X\log\frac{1}{1-\Phi}}\\
&=\frac{1}{s}((1-\Phi)^{-s}-1)(1-\Phi)^{-X}\\
&=\frac{1}{s}((1-\Phi)^{-X-s}-(1-\Phi)^{-X}).
\end{aligned}$$

This is equal to (4), so the theorem is proved. □

References

[1] M. Lassalle, *Quelques conjectures combinatoires relatives à la formule classique de Chu-Vandermonde*, Adv. in Appl. Math. **21**, (1998), 457-472.

[2] M. Lassalle, *Une conjecture en théorie des partitions*, manuscript, math.CO/9901040.

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